The influence of radiative transfer on the propagation of a temperature wave in a stratified diffusing atmosphere

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An approximate solution is presented to the problem of the propagation of a temperature wave through a stratified medium which both diffuses and radiates heat. The solution is a combination of two waves whose relative amplitudes vary with the distance from the lower boundary. Apparent diffusivities computed from phase lag and attenuation coefficients can differ greatly from each other and can vary with height even if the actual diffusivity does not.

An example, using parameters simulating the earth's atmosphere, suggests that analysis of the propagation of the diurnal wave upward from the earth's surface is likely to be inadequate if radiative effects are not considered.

1. Introduction

Rider & Robinson (1951) have shown that, in not uncommon circumstances in the lowest layers of the atmosphere, radiation is comparable in importance to convection in the control of temperature and humidity. A theoretical treatment of the problem of the upward propagation of the diurnal temperature wave should therefore include radiative transfer terms. The only attempt to do so has been that of Brunt (1944, p. 129) who employed an analogy between radiative and diffusive transfer, similar to that introduced earlier by Eddington into astrophysical problems for very great optical depths. This is, however, not suitable for this particular problem. The problem will be re-examined using more acceptable approximations.

Goody (1956) has given approximate equations for a grey-absorbing, stratified atmosphere. While the grey approximation is a poor one for the earth's atmosphere it is valuable, on heuristic grounds, to examine the problem of the propagation of a diurnal temperature wave in a grey absorbing atmosphere in order to discover the essential physical effects of this mode of heat transfer.

Following classical treatments of atmospheric diffusion (Priestley 1959), it will be assumed that turbulent diffusion can be described by a simple conduction equation. Moreover, the turbulent conductivity will be treated as a constant throughout the region under consideration. It is not necessary to dwell upon the inadequacies of this assumption but, provided that they are not forgotten, they should not obscure the role of radiative transfer as a modifying influence, which is the question under consideration here.

The approximate equation governing the radiative flux in a horizontally stratified fluid is (Goody 1956),

$$\frac{d^2F_r}{dz^2} = 4\pi Q\kappa\frac{d\theta}{dz} + 3\kappa^2 F_r, \tag{1}$$

where F_r = radiative heat flux, z = height,
$\pi Q = 4\sigma\theta^3$, σ = Stefan's constant,
θ = temperature, κ = volume absorption coefficient.

It will be assumed that the range of θ is sufficiently small for Q to be considered constant. At the lower boundary ($z = 0$) the boundary condition is

$$\frac{dF_r}{dz} = -2\kappa F_r. \tag{2}$$

The vertical turbulent heat flux is

$$F_t = -k\frac{d\theta}{dz}, \tag{3}$$

where k is the turbulent 'conductivity'. The solution to the problem

$$F = F_r + F_t = \text{const.}, \tag{4}$$

appropriate to a semi-infinite atmosphere whose lower boundary ($z = 0$) is a black body, can be written (Goody 1956)

$$\frac{d\theta}{dz} = -\frac{F}{k}\left\{\frac{1}{1+\chi} + \left(\frac{2\chi}{1+\chi}\right)\frac{\exp[-\sqrt{(1+\chi)}]}{2+\sqrt{[3(1+\chi)]}}\zeta\right\}, \tag{5}$$

where
$$\zeta = \sqrt{3}\,\kappa z, \tag{6}$$

$$\chi = \frac{4\pi Q}{3\kappa k}. \tag{7}$$

The non-dimensional quantities ζ and χ are, respectively, a modified optical depth which takes some account of the diffuse nature of the radiation, and the ratio between Brunt's radiative conductivity and the turbulent conductivity. Equation (5) shows that for $\chi > 1$ a temperature boundary layer can develop.

2. Propagation of a harmonic thermal wave

If the temperature is time-dependent, then in place of (4) we have

$$\frac{\partial F}{\partial z} = -s\frac{\partial \theta}{\partial t}, \tag{8}$$

where s is the heat content of the fluid per cm^3 per degree. From (1), (3), (6) and (8), we find

$$-\left[\frac{\partial^2}{\partial\zeta^2} - 1\right]\frac{\partial\theta}{\partial t} = \frac{4\pi Q\kappa}{s}\frac{\partial^2\theta}{\partial\zeta^2} - \frac{3\kappa^2 k}{s}\left\{\frac{\partial^2}{\partial\zeta^2} - 1\right\}\frac{\partial^2\theta}{\partial\zeta^2}. \tag{9}$$

Steady-state harmonic solutions of the type

$$\theta = \theta_0\, e^{i\omega t}\, e^{-(a+ib)\zeta} \tag{10}$$

will be sought. Transient solutions are not considered and only the time-dependent part of the solution is discussed. Solutions of (4) should be added for completeness, but oscillations about the time mean will be correctly described by (10). Only positive values of a will be allowed so that the amplitude of the wave will tend to zero at great optical distances from the lower boundary. This is the upper boundary condition.

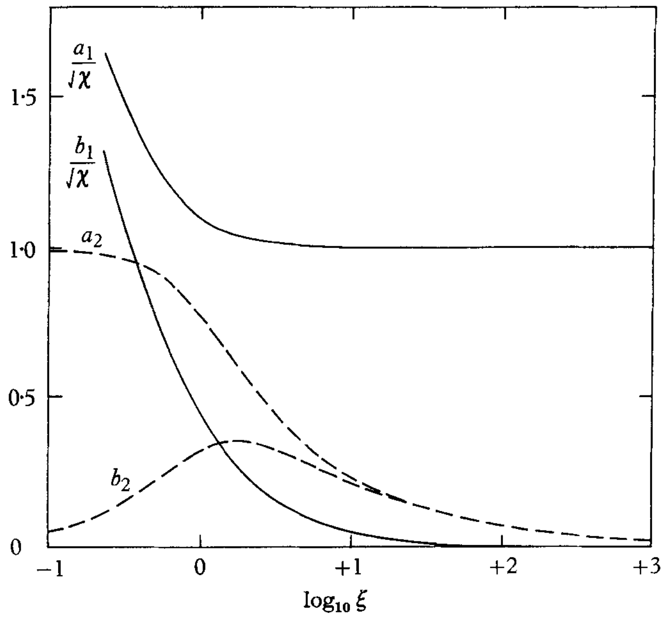

FIGURE 1. Attenuation and phase-lag coefficients for $\chi \gg 1$.

Substituting $\partial/\partial t = i\omega$ in (9) we obtain

$$-i\left[\frac{\partial^2}{\partial\zeta^2}-1\right]\theta = \xi\frac{\partial^2\theta}{\partial\zeta^2}-\frac{\xi}{\chi}\left[\frac{\partial^2}{\partial\zeta^2}-1\right]\frac{\partial^2\theta}{\partial\zeta^2}, \tag{11}$$

where
$$\xi = \frac{4\pi Q\kappa}{s\omega}. \tag{12}$$

Substituting (10) into (11) with

$$a^2-b^2 = \alpha, \quad 2ab = \beta,$$

we find, by equating real and imaginary parts,

$$\beta = \frac{1-\alpha}{\xi+\frac{\xi}{\chi}(1-2\alpha)}, \tag{13}$$

$$\xi^2\alpha\left(1+\frac{1-\alpha}{\chi}\right)\left(1+\frac{1-2\alpha}{\chi}\right)^2 = (1-\alpha)\left(1-\frac{\alpha}{\chi}\right). \tag{14}$$

Equation (14) has only two real roots, both positive, for which β is also positive. Consequently, if a is always positive then b must also be positive. The wave is therefore propagated with a phase *lag* which increases linearly with height.

Since two real roots of (4) exist and since the boundary conditions are sufficient to determine the amplitudes of both solutions, we have the required solution to our problem. However, the roots of (14) are difficult to evaluate except when $\chi \gg 1$ or $\chi \ll 1$. Discussion will be restricted to these two cases, since they suffice to demonstrate the essential physical features of the problem.

The case $\chi \gg 1$

One root of α lies between χ and $1+\chi$ while the other lies between 0 and 1. In these intervals the roots can be located by successive approximation. If subscripts 1 and 2 are used to designate the two solutions, there results to order $1/\chi$,

$$\left.\begin{aligned} a_1^2 &= \frac{\chi}{2}[\surd(1+1/\xi^2)+1], \quad a_2^2 = \frac{1}{2(1+\xi^2)}[\surd(1+\xi^2)+1], \\ b_2^2 &= \frac{\chi}{2}[\surd(1+1/\xi^2)+1], \quad b_2^2 = \frac{1}{2(1+\xi^2)}[\surd(1+\xi^2)-1]. \end{aligned}\right\} \tag{15}$$

These relations are shown graphically in figure 1.

The case $\chi \ll 1$

One root of α now lies between 1 and $1+\chi$ and the other between 0 and χ. To order χ the two roots are given by

$$\left.\begin{aligned} a_1^2 &= 1, \quad a_2^2 = \frac{\chi}{2\xi}, \\ b_1^2 &= \frac{\xi^2}{4}\left(1+\frac{\xi^2}{\chi^2}\right)^{-2}, \quad b_2^2 = \frac{\chi}{2\xi}. \end{aligned}\right\} \tag{16}$$

3. Boundary conditions

The boundary condition (2) is consistent with the approximation used to derive equation (1). Eliminating F_r between (2) and (1) we have, at $\zeta = 0$,

$$\frac{4\pi Q}{\surd 3}\frac{\partial\theta}{\partial\zeta} = \frac{\partial^2 F_r}{\partial\zeta^2} - \frac{\surd 3}{2}\frac{\partial F_r}{\partial\zeta}. \tag{17}$$

From (3), (4), (8) and (10) we have

$$\frac{\surd 3}{2}\frac{\partial F_r}{\partial\zeta} = \frac{3k\kappa}{2}\frac{\partial^2\theta}{\partial\zeta^2} - \frac{is\omega}{2\kappa}\theta, \tag{18}$$

and

$$\frac{\partial^2 F_r}{\partial\zeta^2} = \surd 3\,k\kappa\frac{\partial^3\theta}{\partial\zeta^3} - \frac{is\omega}{\surd 3\,\kappa}\frac{\partial\theta}{\partial\zeta}. \tag{19}$$

Eliminating $\partial F_r/\partial\zeta$ and $\partial^2 F_r/\partial\zeta^2$ from (17), (18) and (19), we find, at $\zeta = 0$,

$$\frac{\surd 3}{2}\left[\frac{\xi}{\chi}\frac{\partial^2\theta}{\partial\zeta^2} - i\theta\right] = \left[\frac{\xi}{\chi}\frac{\partial^3\theta}{\partial\zeta^3} - i\frac{\partial\theta}{\partial\zeta} - \xi\frac{\partial\theta}{\partial\zeta}\right]. \tag{20}$$

This is the lower boundary condition for θ. (10) can now be written

$$\frac{\theta}{\theta_0} = (p_1+iq_1)\,e^{i\omega t}\,e^{-(a_1+ib_1)\zeta} + (p_1+iq_1)\,e^{i\omega t}\,e^{-(a_2+ib_2)\zeta}, \tag{21}$$

where
$$\left.\begin{aligned} q_1+q_2 &= 0, \\ p_1+p_2 &= 1. \end{aligned}\right\} \tag{22}$$
and

Substituting (21) into (20) and equating real and imaginary parts gives two further relationships, which completely determine the solution. There results

$$\left.\begin{aligned} p_1 &= \frac{R_2(R_2-R_1)+I_2(I_2-I_1)}{(R_1-R_2)^2+(I_1-I_2)^2}, \\ q_1 &= \frac{R_2 I_1 - R_1 I_2}{(R_1-R_2)^2+(I_1-I_2)^2}, \end{aligned}\right\} \tag{23}$$

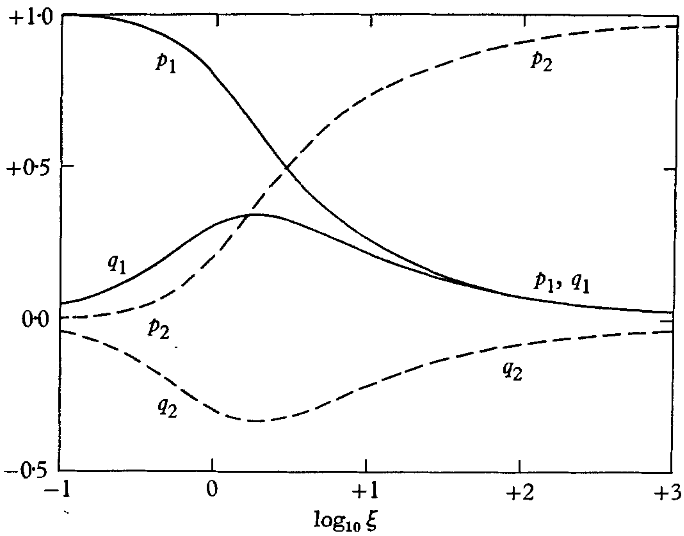

FIGURE 2. Amplitudes of the wave components for $\chi \gg 1$.

	p_1	p_2	q_1	q_2
$\xi \to 0$	$1-0{\cdot}340\xi^2$	$0{\cdot}340\xi$	$0{\cdot}464\xi$	$-0{\cdot}464\xi$
$\xi \to \infty$	$0{\cdot}94\xi^{-\frac{1}{2}}$	$1-0{\cdot}94\xi^{-\frac{1}{2}}$	$0{\cdot}94\xi^{-\frac{1}{2}}$	$-0{\cdot}94\xi^{-\frac{1}{2}}$

TABLE 1. Amplitudes for $\chi \gg 1$ and for ξ large and small.

where I_1, I_2 and R_1, R_2 are the two alternative values of

$$\left.\begin{aligned} R &= \frac{\sqrt{3}}{2}\frac{\xi}{\chi}\alpha + b\left(1-\frac{\xi}{\chi}\beta\right) + a\xi\left(\frac{\alpha}{\chi}-1\right), \\ I &= \left(\frac{\sqrt{3}}{2}+a\right)\left(\frac{\xi}{\chi}\beta-1\right) + \xi b\left(\frac{\alpha}{\chi}-1\right). \end{aligned}\right\} \tag{24}$$
and

Since we have already restricted our attention to very large and very small χ, we will not determine the amplitudes for all values of the variable. For $\chi \ll 1$, the solution, to order χ, is

$$p_1 = q_1 = q_2 = 0, \quad p_2 = 1. \tag{25}$$

For $\chi \gg 1$ p_1 and q_1 are functions of ξ only. The values of p_1, p_2, q_1 and q_2 are shown graphically for all values of ξ in figure 2. Table 1 gives algebraic forms for the limits $\xi \to 0$ and $\xi \to \infty$.

4. Discussion of the results

For $\chi \ll 1$ radiative effects are unimportant, since the (p_1, q_1) solution does not come into consideration and we have, from (16),

$$\zeta a_2 = \zeta b_2 = z\sqrt{\frac{s\omega}{2k}}. \tag{26}$$

(26) is the solution for a non-radiating medium. The condition $\chi \ll 1$ is therefore a sufficient condition for neglecting radiative effects.

ξ	$\xi\sqrt{2\chi}$	$\dfrac{\chi}{2\xi a_1^2}$	$\dfrac{\chi}{2\xi b_1^2}$	$\dfrac{1}{2\xi a_2^2}$	$\dfrac{1}{2\xi b_2^2}$
$\frac{1}{4}$	1·88	0·78	1·28	2·10	141·6
$\frac{1}{2}$	1·67	0·62	1·62	1·19	21·22
$1/\sqrt{2}$	1·43	0·52	1·93	0·95	9·78
1	1·21	0·41	2·42	0·83	4·82
$\sqrt{2}$	0·74	0·32	3·13	0·78	2·90
2	0·22	0·24	4·49	0·77	2·02
4	—	—	—	0·83	1·36
8	—	—	—	0·90	1·15
10	—	—	—	0·91	1·11
30	—	—	—	0·97	1·02
100	—	—	—	1·00	1·01

TABLE 2. Numerical values for some important functions of ξ, for $\chi \gg 1$. Blanks indicate that the quantity cannot be observed in practice.

The situation is more interesting if χ is large. Both solutions can be important and they have different phase lag and attenuation coefficients. For $\xi < 2{\cdot}8$ the amplitude $[\sqrt{(p^2+q^2)}]$ of the (p_1, q_1) solution exceeds that of the (p_2, q_2) solution, while for $\xi > 2{\cdot}8$ the position is reversed. Since the attenuation coefficients are not the same for the two solutions the relative amplitudes change with height. From figure 1, $a_1 \gg a_2$ if $\chi \gg 1$ for all ξ. Thus if the (p_2, q_2) solution is the more important at one level it will be the more important for all higher levels. However, if the (p_1, q_1) solution is more important at some level there is always a higher level at which it will be overtaken by the (p_2, q_2) solution and the character of the wave will then change. The changeover point (ζ_c) will be determined by

$$(p_1^2+q_1^2)^{\frac{1}{2}} \exp[-a_1\zeta_c] = (p_2^2+q_2^2)^{\frac{1}{2}} \exp[-a_2\zeta_c]. \tag{27}$$

Since for $\chi \gg 1$, $a_1 \gg a_2$,

$$\sqrt{(2\chi)}\,\zeta_c \doteqdot \frac{1}{\sqrt{(2/\chi)}\,a_1} \ln\frac{p_1^2+q_1^2}{p_2^2+q_2^2}. \tag{28}$$

The right-hand side of equation (28) is a function of ξ only and some values are given in table 2.

The general solution for $\chi \gg 1$ does not lend itself readily to description. One course is to discuss the effective conductivities (k (phase) and k (amp)) based upon the gradient of phase (Φ) and the gradient of the logarithm of the amplitude ($\ln|\theta|$)

$$\frac{d\Phi}{dz} = -\sqrt{\frac{s\omega}{2k\,(\text{phase})}}, \tag{29}$$

$$\frac{d\ln|\theta|}{dz} = -\sqrt{\frac{s\omega}{2k\,(\text{amp})}}. \tag{30}$$

According to (26), for $\chi \ll 1$ equations (29) and (30) will yield

$$k\,(\text{amp}) = k\,(\text{phase}) = k$$

in an atmosphere whose conductivity varies neither with time nor height. Under other circumstances (29) and (30) are formal operations, with no particular physical meaning, which are often used to describe observed temperature profiles (Best 1935; Priestley 1959).

If either the (p_1, q_1) or the (p_2, q_2) solution dominates, then from (10),

$$\frac{d\ln|\theta|}{dz} = -a\frac{d\zeta}{dz} = -\sqrt{3}\,a\kappa, \tag{31}$$

and

$$\frac{d\Phi}{dz} = -b\frac{d\zeta}{dz} = -\sqrt{3}\,b\kappa, \tag{32}$$

where a stands for a_1 or a_2 and b for b_1 or b_2. Thus,

$$\frac{k\,(\text{amp})}{k} = \frac{s\omega}{6a^2k\kappa^2} = \frac{\chi}{2a^2\xi}, \tag{33}$$

$$\frac{k\,(\text{phase})}{k} = \frac{s\omega}{6b^2k\kappa^2} = \frac{\chi}{2b^2\xi}. \tag{34}$$

Table 2 shows values of

$$\frac{\chi}{2\xi a_1^2},\quad \frac{\chi}{2\xi b_1^2},\quad \frac{1}{2\xi a_2^2},\quad \frac{1}{2\xi b_2^2}.$$

The Eddington approximation as used by Brunt (1944) gives

$$\frac{k\,(\text{amp})}{k} = \frac{k\,(\text{phase})}{k} = (\chi+1). \tag{35}$$

The results shown in table 2 demonstrate that for $\chi \gg 1$, (35) is correct in the limit $\xi \to \infty$.

5. Application to the atmosphere

Two definitions of the mean absorption coefficient, discussed by Goody (1953), are

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{d\theta}\,d\nu}{\int_0^\infty \frac{dB_\nu}{d\theta}\,d\nu}, \tag{36}$$

and

$$\kappa_P = \frac{\int_0^\infty \frac{dB_\nu}{d\theta}\,d\nu}{\int_0^\infty \frac{dB_\nu}{d\theta}\,d\nu}, \tag{37}$$

where

ν = frequency,

κ_R = Rosseland's mean (Kourganoff 1952, p. 237),

κ_P = Planck's mean (slightly modified from the definition of Michard 1949).

Contributions to κ_P come mainly from the strongly absorbing line centres, about which we have comparatively reliable information. Since the integral in equation (37) embraces each line completely, its value should not vary with the air pressure, since the integrated line absorption coefficient is independent of this parameter. Consideration of the data of Cowling (1950), interpreted with the help of the statistical model, and the data of Kaplan (1950) lead to

$$\kappa_P = (203\rho_{H_2O} + 87\rho_{CO_2})\,\text{cm}^{-1}, \tag{38}$$

where ρ_{H_2O} and ρ_{CO_2} are densities of water vapour and carbon dioxide in g cm^{-3}.

Pressure (mb)	$\rho_{CO_2}^{(1)}$ (g cm^{-3})	$\rho_{H_2O}^{(2)}$ (g cm^{-3})	$\kappa_R^{(3)}$ (cm^{-1})	$\kappa_P^{(4)}$ (cm^{-1})
1000	5×10^{-7}	7×10^{-6}	1.4×10^{-6}	1.4×10^{-3}
100	5×10^{-8}	2×10^{-10}	4×10^{-12}	4×10^{-6}
10	5×10^{-9}	2×10^{-11}	4×10^{-14}	4×10^{-7}

(1) Assuming 2.6×10^{-4} parts of CO_2 by volume (Gluckauf 1944).
(2) Assuming 2×10^{-6} parts of H_2O by volume at 100 and 10 mb (Murgatroyd, Goldsmith & Hollings 1955), and 10^{-2} parts by volume at 1000 mb.
(3) H_2O contribution only.
(4) H_2O and CO_2 contributions.

TABLE 3. Mean absorption coefficients.

The integral in equation (36) is dominated by the small absorption coefficients between major bands. For water vapour the main contribution will come from the 10μ window. New values of the absorption coefficient in this region of the spectrum at atmospheric pressure have recently been measured (Roach & Goody 1958). The effect of the strong 15μ band of carbon dioxide is to reduce slightly the effect of water valour by increasing κ_ν over a narrow region. For water vapour alone

$$\kappa_R = 0.21\rho_{H_2O}\left(\frac{p}{1000}\right)\text{cm}^{-1}, \tag{39}$$

where p is the pressure in millibars. The linear pressure factor is assumed, since we are here concerned with contributions from the far wings of pressure-broadened lines.

We will limit our attention to ground level and, in order to have an idea of the magnitudes involved, we will take the harmonic mean of κ_R and κ_P,

$$\kappa = \sqrt{(\kappa_R\kappa_P)} \simeq 4.5 \times 10^{-5}\,\text{cm}^{-1}.$$

Best (1935) gives values for k/s computed from the amplitude and phase of the observed diurnal temperature wave near the ground which vary from 2 cm^2 sec^{-1} near z = 10 cm to 5000 cm^2 sec^{-1} near $z = 10^3$ cm. We may adopt

$$k/s = 10^2\,\text{cm}^2\,\text{sec}^{-1}$$

as characteristic of the region. In addition assume:

$$\theta = 290^\circ \mathrm{K},$$
$$\sigma = 5{\cdot}74 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,sec^{-1}\,{}^\circ K^{-4}},$$
$$\pi Q = 5{\cdot}6 \times 10^{3}\,\mathrm{erg\,cm^{-2}\,sec^{-1}\,{}^\circ K^{-1}},$$
$$s = 1{\cdot}36 \times 10^{4}\,\mathrm{erg\,cm^{-3}\,{}^\circ K^{-1}},$$
$$\omega\,(\text{diurnal}) = 7{\cdot}3 \times 10^{-5}\,\mathrm{sec^{-1}}.$$

With these values

$$\chi = 120, \quad \xi = 1{\cdot}0.$$

From Table 2, for $\xi = 1$

$$\zeta_c \sqrt{2\chi} = \sqrt{(6\chi)}\,\kappa z_c = 1{\cdot}21.$$

Hence,

$$z_c = 1 \times 10^3\,\mathrm{cm}.$$

Below this critical level, where the (p_1, q_1) solution will predominate,

$$\frac{k\,(\text{amp})}{s} = 4{\cdot}1 \times 10\,\mathrm{cm^2\,sec^{-1}},$$

$$\frac{k\,(\text{phase})}{s} = 2{\cdot}42 \times 10^2\,\mathrm{cm^2\,sec^{-1}}.$$

Above 10 m, the (p_2, q_2) solution will predominate and

$$\frac{k\,(\text{amp})}{s} = 1 \times 10^4\,\mathrm{cm^2\,sec^{-1}},$$

$$\frac{k\,(\text{phase})}{s} = 5{\cdot}7 \times 10^4\,\mathrm{cm^2\,sec^{-1}}.$$

Brunt's solution for this problem would be

$$k\,(\text{phase})/s = k\,(\text{amp})/s = 1{\cdot}2 \times 10^4.$$

6. Conclusions

There is a superficial resemblance between the behaviour of the effective conductivity predicted from this simple theory and that observed in the earth's atmosphere. k is expected to increase rapidly with height, and to differ according to whether it is computed from the amplitude or phase. Moreover, since the diffusion of momentum is not directly affected by radiative transfer, its effective coefficient will differ from that for heat. All these features are observed in the earth's atmosphere and they are usually explained on a purely dynamical basis. This investigation does not invalidate such explanations, but it does indicate a possible alternative mechanism which may play some role. A satisfactory treatment should consider all modes of heat transfer more realistically than in this paper. The crude approximations used here may, however, not completely obscure the broad physical features of the problem.

Deep in the interior of a star, away from boundaries or any other imposed discontinuities, the Eddington approximation may be justified. Very different conditions prevail in a planetary atmosphere, however, and the numerical results show clearly how unsatisfactory the approximation can be.

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